Manifestations the hidden symmetry of Coulomb problem in the relativistic quantum mechanics - from Pauli to Dirac electron

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Abstract

The theorem known from Pauli equation about operators that anticommute with Dirac's K-operator is generalized to the Dirac equation. By means of this theorem the operator is constructed which governs the hidden symmetry in relativistic Coulomb problem (Dirac equation). It is proved that this operator coincides with the familiar Johnson-Lippmann one and is intimately connected to the famous Laplace-Runge-Lenz (LRL) vector. Our derivation is very simple and informative. It does not require a longtime and tedious calculations, as is offten underlined in most papers.

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It is well known that the Coulomb Problem has additional dynamical symmetry in classical mechanics as well as in non-relativistic quantum mechanics. This symmetry guarantees the classical motion on the closed orbits. At the same time, it is responsible to the "accidental" degeneracy of hydrogen atom spectrum. The nature of this phenomena was explained by V.Fock [1], V. Bargman [2], W.Pauli [3] and others many years ago.

There are no closed orbits in relativistic Kepler problem, however. Recall that Sommerfeld [4] obtained his famous energy levels of the hydrogen atom by transforming to a rotating coordinate system in which the relativistic precession (the familiar "rosette" motion) was eliminated and closed orbits occur. This means that there is still some residual hidden symmetry of the Kepler problem in relativistic mechanics as well.

This consideration is enhanced by recent investigations. It was shown [5], that the Dirac equation still have some "hidden" symmetry, which ultimately provides a certain algebra (so called, Witten algebra), that gives a quantum-mechanical supersymmetry of this problem.

In other words, the additional symmetry (conservation of the Laplace-Runge-Lenz, hereafter, LRL vector) transforms into a certain supersymmetry in microworld (hydrogen atom), which controls the degeneracy of the hydrogen spectra in the Dirac equation.

Together with bosonic degrees of freedom supersymmetry requires existence of fermionic degrees of freedom as well. In this purpose it is easier to discuss the motion in a Coulomb field of a Pauli particle (nonrelativistic spin-1/2 particle with kinematically independent spin). This simplicity results from using the spin operator $\vec{\sigma}$, together with the two vector operators $\vec{l}$ (angular momentum) and $\vec{A}$ (LRL vector). With the help of spin operator the Dirac nonrelativistic operator's analogue is constructed, $K_p = -(2\vec{\sigma}\vec{l} + 1)$. It commutes with the Pauli Hamiltonian and allows introduction of Z_2 graduation in the Hilbert space by making use of parity operator $P_k = K_p/|k|$ and the physical states (operators) are divided into even and odd ones.

To find odd operators certain theorem is used [6], which now we will generalize below for the Dirac case.

The Dirac Hamiltonian in Coulomb potential has the form

$$H = \vec{\alpha}\vec{p} + \beta m - \frac{a}{r} \ , \quad a \equiv Ze^2 = Z\alpha \ . \tag{1}$$

It is known, that the total momentum operator $\vec{J} = \vec{l} + \vec{\Sigma}/2$ commutes with H, where $\vec{\Sigma}$ is the spin matrix

$$\vec{\Sigma} = \gamma^5 \vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \ .$$

It is connected to the usual Dirac's matrices by relation $\vec{\Sigma} = \gamma^5\vec{\alpha}$. It is the Dirac's K operator

$$K = \beta\left(\vec{\Sigma}\vec{l} + 1\right) \ , \tag{2}$$

which commutes with H. It has the following properties

$$\beta K = K\beta \ , \qquad \gamma^5 K = -K\gamma^5 \ . \tag{3}$$

The spectrum of Hamiltonian (1) is degenerate with respect to two signs of the K's eigenvalues, $\pm k$ [6]. It is evident that the operator, that interchange signs of the k, must be anticommuting with K. If such anticommuting operator at the same time would be commuting with Hamiltonian, there arises the additional symmetry, which is exactly Witten's superalgebra.

Let us generalize the above mentioned theorem from Pauli to Dirac electron.

Theorem:

Suppose $\vec{V}$ is a vector with respect to the orbital angular momentum $\vec{l}$ that is also perpendicular to $\vec{l}$, i.e.,

$$\vec{l} \times \vec{V} + \vec{V} \times \vec{l} = 2i\vec{V} \ , \qquad \vec{l}\vec{V} = \vec{V}\vec{l} = 0 \ .$$

then K anticommutes with the $\vec{J}$ scalar, $\vec{\Sigma}\vec{V}$.

Proof: Let us consider a product $(\vec{\Sigma}\vec{l})(\vec{\Sigma}\vec{V})$. Exploiting the properties of Dirac matrices, one can establish that

$$(\vec{\Sigma}\vec{l})(\vec{\Sigma}\vec{V}) = (\vec{l}\vec{V}) - i(\vec{\Sigma},\ \vec{l} \times \vec{V}) = i(\vec{\Sigma},\ 2i\vec{V} - \vec{V} \times \vec{l}) = -2\vec{\Sigma}\vec{V} - i\vec{\Sigma}(\vec{V} \times \vec{l}) \ .$$

Therefore

$$\left(\vec{\Sigma}\,\vec{l} + 1\right)\left(\vec{\Sigma}\,\vec{V}\right) = -\vec{\Sigma}\,\vec{V} - i\vec{\Sigma}(\vec{V} \times \vec{l}) \ . \tag{4}$$

Now consider the same product in reversed order

$$(\vec{\Sigma}\vec{V})(\vec{\Sigma}\vec{l}) = \vec{V}\vec{l} + i\vec{\Sigma}(\vec{V} \times \vec{l}) = i\vec{\Sigma}(\vec{V} \times \vec{l}) \ .$$

Hence

$$\vec{\Sigma}\vec{V}(\vec{\Sigma}\vec{l} + 1) = \vec{\Sigma}\vec{V} + (\vec{\Sigma}\vec{V})(\vec{\Sigma}\vec{l}) = \vec{\Sigma}\vec{V} + i\vec{\Sigma}(\vec{V} \times \vec{l}) = -(\vec{\Sigma}\vec{l} + 1)(\vec{\Sigma}\vec{V}) \ .$$

In the last step we made use of equation (4). Therefore we obtain

$$\{\vec{\Sigma}\vec{l} + 1,\ \vec{\Sigma}\vec{V}\} = 0 \ . \tag{5}$$

Now it follows finally, that

$$K(\vec{\Sigma}\vec{V}) = -(\vec{\Sigma}\vec{V})K \ . \tag{6}$$

It is evident that the class of anticommuting with K (or, K-odd) operators is not confined by these operators only - any operator of type $\hat{O}(\vec{\Sigma}\vec{V})$, where $\hat{O}$ is commuting with K, but otherwise arbitrary, also is K-odd.

Let mention, that in the framework of constraints of above theorem, the following very useful relation takes place

$$K\left(\vec{\Sigma}\vec{V}\right) = -i\beta\left(\vec{\Sigma}\ ,\ \frac{1}{2}\left[\vec{V}\times\vec{l} - \vec{l}\times\vec{V}\right]\right)\ . \qquad (7)$$

One can see that the antisymmetrized vector product, familiar to LRL vector appears on the right-hand-side of this relation.

Important special cases, resulting from the above theorem include $\vec{V} = \hat{\vec{r}}$ (unit radial vector), $\vec{V} = \vec{p}$ (linear momentum) and $\vec{V} = \vec{A}$ (LRL vector), which has the following form

$$\vec{A} = \hat{\vec{r}} - \frac{i}{2ma}\left[\vec{p}\times\vec{l} - \vec{l}\times\vec{p}\right]\ . \qquad (8)$$

According to (7), there appears one relation between these three odd operators

$$\vec{\Sigma}\vec{A} = \vec{\Sigma}\hat{\vec{r}} + \frac{i}{ma}\beta K\left(\vec{\Sigma}\vec{p}\right)\ . \qquad (9)$$

As far as, $[\beta\ ,\ K] = 0$, it follows that $\left\{K\ ,\ \beta K\left(\vec{\Sigma}\vec{p}\right)\right\} = 0$ and $K\left(\vec{\Sigma}\vec{p}\right)$ can be used as a permissible K-odd operator.

Our purpose is to construct such combination of K-odd operators, which would be commuting with Dirac Hamiltonian. We can solve this task by step by step. As a first trial expression let consider the following operator

$$A_1 = x_1\left(\vec{\Sigma}\hat{\vec{r}}\right) + ix_2K\left(\vec{\Sigma}\vec{p}\right)\ .$$

Here the coefficients are chosen in such a way, that A_1 be Hermitian, when $x_1\ ,\ x_2$ are arbitrary real numbers. These numbers must be determined from the requirement of commuting with H. Let calculate

$$[A_1, H] = x_1\left[\left(\vec{\Sigma}\hat{\vec{r}}\right), H\right] + ix_2\left[K\left(\vec{\Sigma}\vec{p}\right), H\right]\ .$$

Appearing here commutators can be calculated easily. The result is

$$[A_1, H] = x_1\frac{2i}{r}\beta K\gamma^5 - x_2\frac{a}{r^2}K\left(\vec{\Sigma}\vec{r}\right)\ .$$

One can see, that the first term in right-hand side is antidiagonal, while the second term is diagonal. So this expression never becomes vanishing for ordinary real numbers $x_1\ ,\ x_2$. Therefore we must perform the second step: one has to include new odd structure, which appeared on the right-hand side of above expression. Hence, we are faced to the new trial operator

$$A_2 = x_1\left(\vec{\Sigma}\hat{\vec{r}}\right) + ix_2K\left(\vec{\Sigma}\vec{p}\right) + ix_3K\gamma^5 f(r)\ . \qquad (10)$$

Here $f(r)$ is an arbitrary scalar function to be determined. Let calculate new commutator. We have

$$[A_2, H] = x_1 \frac{2i}{r} \beta K \gamma^5 - x_2 \frac{a}{r^2} K \left(\vec{\Sigma}\hat{\vec{r}}\right) - x_3 f'(r) K \left(\vec{\Sigma}\hat{\vec{r}}\right) - i x_3 2m\beta K \gamma^5 f(r) \ .$$

Grouping diagonal and antidiagonal matrices separately and equating this expression to zero, we obtain equation

$$K \left(\vec{\Sigma}\hat{\vec{r}}\right) \left(\frac{a}{r^2} x_2 + x_3 f'(r)\right) + 2i\beta K \gamma^5 \left(\frac{1}{r} x_1 - m x_3 f(r)\right) = 0 \ .$$

This equation is to be satisfied, if diagonal and antidiagonal terms become zero separately, i.e.,

$$\frac{a}{r^2} x_2 = -x_3 f'(r) \quad , \quad \frac{1}{r} x_1 = m x_3 f(r) \ .$$

First of all, let us integrate the second equation in the interval $(r \ , \ \infty)$. It follows

$$x_3 f(r) = -\frac{a}{r} x_2 \ .$$

Accounting this in the first equation, we have

$$x_2 = -\frac{1}{ma} x_1 \ .$$

Then we obtain also

$$x_3 f(r) = -\frac{1}{mr} x_1 \ .$$

Therefore finally we have derived the following operator which commutes with Dirac Hamiltonian

$$A_2 = x_1 \left\{ \left(\vec{\Sigma}\hat{\vec{r}}\right) - \frac{i}{ma} K \left(\vec{\Sigma}\vec{p}\right) + \frac{i}{m} K \gamma^5 \frac{1}{r} \right\} \ . \qquad (11)$$

It is K-odd, in accord with above theorem.

If we turn to usual $\vec{\alpha}$ matrices using the relation $\vec{\Sigma} = \gamma^5 \vec{\alpha}$ and taking into account the expression (1) of the Dirac Hamiltonian, A_2 can be reduced to the more familiar form (x_1, as unessential common factor, may be dropped)

$$A_2 = \gamma^5 \left\{ \vec{\alpha}\hat{\vec{r}} - \frac{i}{ma} K \gamma^5 \left(H - \beta m\right) \right\} \ .$$

This expression is nothing but the Johnson-Lippmann operator that was introduced by these authors in a very brief abstract in 1950. As to more detailed settle, by our knowledge, it had not been published neither then nor after.

It seems that our derivation of this operator is rather easy and transparent. Moreover, in parallel, we have shown the commutativity of this operator with the Dirac Hamiltonian.

In order to clear up its physical meaning, remark, that equation (11) may be rewritten in the form

$$A_2 = \vec{\Sigma} \left(\hat{\vec{r}} - \frac{i}{2ma} \beta \left[\vec{p} \times \vec{l} - \vec{l} \times \vec{p} \right] \right) + \frac{i}{mr} K \gamma^5 \ .$$

It is evident, that in nonrelativistic limit, when $\beta \to 1$ and $\gamma^5 \to 0$, this operator reduces to the projection of LRL vector on the electron spin direction, $A_2 \to \vec{\Sigma}\vec{A}$ or because of $\vec{l}\vec{A} = 0$ it is a projection on the total $\vec{J}$ momentum direction.

After this it is clear that the Witten algebra can be derived by identifying supercharges as [5]

$$Q_1 = A \ , \quad Q_2 = i\frac{AK}{k} \ .$$

It follows that

$$\{Q_1, Q_2\} = 0 \ , \quad \text{and} \quad Q_1^2 = Q_2^2 = A^2 \ .$$

This last factor may be identified as a Witten Hamiltonian (N=2 supersymmetry).

As for spectrum, it is easy to show that the following relation occurs [5]

$$A^2 = 1 + \left(\frac{K}{a}\right)^2 \left(\frac{H^2}{m^2} - 1\right) \ .$$

Because all operators entered here commute with each other, we can replace them by their eigenvalues. Therefore we obtain energy spectrum pure algebraically. In this respect it is worthwhile to note full analogy with the classical mechanics, where closed orbits were derived by calculating the square of the LRL vector without solving the differential equations of motion [8].

In conclusion let underline once more that the degeneracy of spectrum relative to interchange $k \to -k$, is connected to the existence of conserved Johnson-Lippmann operator, which takes its origin from the Laplace-Runge-Lenz vector. It is also remarkable that the same symmetry is responsible for absence of the Lamb-shift in this problem. Inclusion the Lamb-shift terms into the Dirac Hamiltonian spoils commutativity of A with H, and consequently, above mentioned supersymmetry.

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